Central Limit Theorem: Let $X_1, X_2, \cdots, X_n$ be i.i.d random variables with mean $\mathrm{E}[X_i] = \mu$, and variance $\mathrm{Var}[X_i] = \sigma^2$, then

$$Z_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \to N(0,1)$$

Proof. Suppose Y is a random variable with $\mathrm{E}[Y] = 0$ and $\mathrm{Var}[Y] = 1$. Then

$$\Phi_Y(t) = E[e^{tY}] = 1 + \frac{t^2}{2} + o(t^2)$$

Let $Y_i = \frac{X_i - \mu}{\sigma}$, then $\mathrm{E}[Y_i] = 0$ and $\mathrm{Var}[Y_i] = 1$, we have

$$\begin{aligned}\Phi_{Z_n}(t) &= E\left[e^{t\left[\frac{X_1+\cdots+X_n-n\mu}{\sigma\sqrt{n}}\right]}\right] \\ &= E\left[e^{\frac{t}{\sqrt{n}}[Y_1+Y_2\cdots+Y_n]}\right] \\ &= E\left[e^{\frac{t}{\sqrt{n}}Y_1}\right]\cdots E\left[e^{\frac{t}{\sqrt{n}}Y_n}\right] \\ &= \left[\Phi_{\mathrm{Y}}\left(\frac{t}{\sqrt{n}}\right)\right]^n \\ &= \left[1 + \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right]^n \quad \to \quad e^{\frac{t^2}{2}}\end{aligned}$$

Therefore, $Z_n \sim N(0,1)$.

Stirling's Formula: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

First Proof. Suppose that the random variable X is Poisson with mean $\mathrm{E}[X] = n,$ and variance $\mathrm{Var}[X] = n,$ then we have

$$\Pr\{X = n\} = \frac{n^n e^{-n}}{n!}$$

Consider the random variable X as the sum of n independent and identical Poisson random variables X_i defined as follows:

$$X = X_1 + X_2 + \cdots + X_n$$

where each X_i is a Poisson random variable with mean 1 and variance 1. From the central limit theorem, we have

$$\frac{X_1 + X_2 + \cdots + X_n - n}{\sqrt{n}} \sim N(0,1)$$

Therefore,

$$Pr\{X = n\} = Pr\left\{-\frac{1}{2\sqrt{n}} \le \frac{X_1 + X_2 + \cdots + X_n - n}{\sqrt{n}} \le \frac{1}{2\sqrt{n}}\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\frac{1}{2\sqrt{n}}}^{\frac{1}{2\sqrt{n}}} e^{-\frac{x^2}{2}}dx = \frac{e^{-\frac{\theta^2}{2}}}{\sqrt{2\pi n}}, \quad \left(\theta \in \left[-\frac{1}{2\sqrt{n}}, \frac{1}{2\sqrt{n}}\right]\right)$$

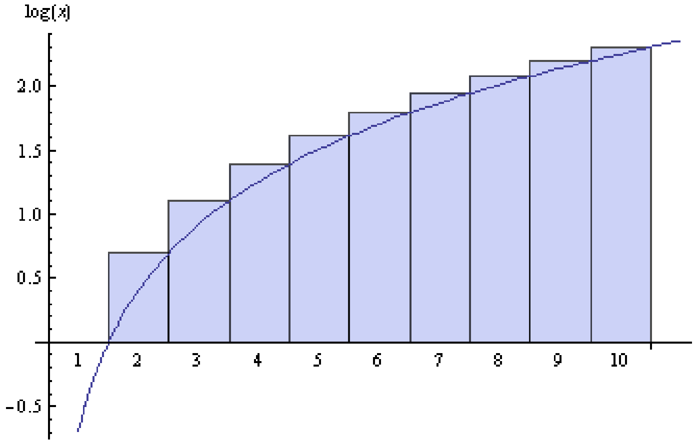
Since $\theta \to 0$ as $n \to \infty,$ hence

$$\frac{n^n e^{-n}}{n!} \sim \frac{1}{\sqrt{2\pi n}}$$

$$\to \quad n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Second Proof. Take logarithm of $n!$, we have

$$\operatorname{Log} n! = \operatorname{Log} 1 + \operatorname{Log} 2 + \cdots \operatorname{Log} n$$



Let I_n denote the area under the curve $\operatorname{Log} x$, we have

$$I_n = \int_1^n \operatorname{Log} x\, dx = n \operatorname{Log} n - n + 1$$

Upper limit (trapezoids):

$$I_n > \frac{1}{2}\operatorname{Log} 1 + \operatorname{Log} 2 + \operatorname{Log} 3 + \cdots + \frac{1}{2}\operatorname{Log} n$$

$$\operatorname{Log} n! < \left(n + \frac{1}{2}\right)\operatorname{Log} n - n + 1$$

Lower limit (middle point):

$$I_n < \frac{1}{8} + \operatorname{Log} 2 + \operatorname{Log} 3 + \cdots + \frac{1}{2}\operatorname{Log} n$$

$$\operatorname{Log} n! > \left(n + \frac{1}{2}\right)\operatorname{Log} n - n + \frac{7}{8}$$

Let $a_n = \operatorname{Log} n! - \left(n + \frac{1}{2}\right)\operatorname{Log} n + n$. We have

$$\frac{7}{8} < a_n < 1$$

To show that $\{a_n\}$ converges, we need to prove that it's monotonic. Use Taylor series expansion, we have

$$\operatorname{Log}\frac{1+t}{1-t} = \operatorname{Log}(1+t) - \operatorname{Log}(1-t) = 2\sum_{k=0}^{\infty}\frac{1}{2k+1}t^{2k+1}$$

Substitute t by $\frac{1}{2n+1}$, we have

$$\operatorname{Log}\frac{n+1}{n} = 2\sum_{k=0}^{\infty}\frac{1}{2k+1}\left(\frac{1}{2n+1}\right)^{2k+1}$$

Hence,

$$a_n - a_{n+1} = \frac{1}{2}(2n+1)\operatorname{Log}\frac{n+1}{n} - 1 = \sum_{k=1}^{\infty}\frac{1}{2k+1}\left(\frac{1}{2n+1}\right)^{2k} > 0$$

Therefore, the sequence $\{a_n\}$ is monotonically decreasing and convergent. Assume that $n! = C\left(\frac{n}{e}\right)^n\sqrt{n}$. We need the following Wallis formula to determine the constant C.

$$S_n = \int_0^{\frac{\pi}{2}} \cos^n x\, dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^{n-2} x\,(1-\sin^2 x)dx$$

$$= S_{n-2} + \int_0^{\frac{\pi}{2}} \cos^{n-2} x\,(-\sin x)\sin x\, dx$$

$$= S_{n-2} + \left.\frac{\cos^{n-1} x \sin x}{n-1}\right|_0^{\frac{\pi}{2}} - \frac{1}{n-1}\int_0^{\frac{\pi}{2}} \cos^n x\, dx$$

$$= S_{n-2} - \frac{1}{n-1}S_n \quad \Longrightarrow \quad \frac{S_n}{S_{n-2}} = \frac{n-1}{n}$$

Since $S_0 = \frac{\pi}{2}$, and $S_1 = 1$, we have

$$S_{2n} = \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2}$$

$$S_{2n-1} = \frac{2n-2}{2n-1}\cdot\frac{2n-4}{2n-3}\cdots\frac{2}{3}\cdot 1$$

$$\Longrightarrow \quad \frac{S_{2n}}{S_{2n-1}} = \pi n\left[\frac{(2n-1)(2n-3)\cdots 3\cdot 1}{2n(2n-2)\cdots 4\cdot 2}\right]^2$$

Because $\{S_n\}$ is monotonically decreasing, and $\frac{S_n}{S_{n-2}} = \frac{n-1}{n} \sim 1$. We have $\frac{S_{2n}}{S_{2n-1}} \sim 1$. Therefore

$$\sqrt{\pi n}\frac{(2n-1)(2n-3)\cdots 3\cdot 1}{2n(2n-2)\cdots 4\cdot 2} \sim 1$$

Multiply the denominator to both nominator and denominator, we have

$$\sqrt{\pi n}\frac{(2n)!}{(2^n n!)^2} \sim 1$$

Substitute $n!$ by $\left(\frac{n}{e}\right)^n\sqrt{n}\cdot C$, we have

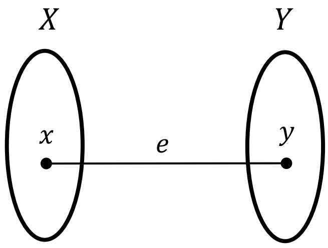
$$\sqrt{\pi n}\frac{\left(\frac{2n}{e}\right)^{2n}\sqrt{2n}\cdot C}{\left(2^n\left(\frac{n}{e}\right)^n\sqrt{n}\cdot C\right)^2} \sim 1 \quad \Longrightarrow \quad C \sim \sqrt{2\pi}$$

Hence, the Stirling's formula $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ is established.

Hall's Theorem: A bipartite graph $\mathrm{G}(X, Y, E)$ contains a matching of X if and only if $|N(S)| \geq S, \quad \forall\, S \subseteq X$.

Proof. We prove the theorem by induction on $|X|$. For $|X| = 1$, the assertion is true. Now, let

$|X(G)| = n\ (n \geq 2)$, and assume Hall's theorem holds for all $|X| < n$.
Case 1: $|N_G(S)| \geq |S| + 1, \quad \forall\, S \subset X$



Take an edge $e = xy$, $x \in X$, $y \in Y$. Let

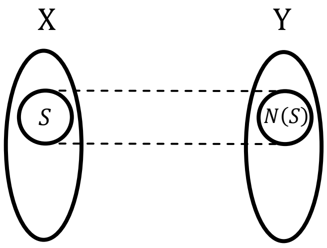
$$G' = G - \{x, y\}$$

Then the condition satisfies:

$$|N_{G'}(S)| \geq S, \quad \forall\, S \subseteq X - \{x\}$$

Because $|X(G')| = |X - \{x\}| = n - 1 < n$, by the induction hypothesis, G' contains a matching of $X - \{x\}$. Together with the edge e, this yields a matching of X.

Case 2: $|N_G(S)| = S, \quad \exists\, S \subset X$



Let

$$G' = S \cup N_G(S)$$

By the induction hypothesis, G' contains a matching M_1 of S. We show that $G - G'$ satisfies the marriage condition by contradiction. Suppose that

$$|N_{G-G'}(S')| < |S'|, \quad \exists S' \subseteq X - S$$

We have

$$|N_G(S' \cup S)| = |N_G(S')| + |N_G(S)| - |N_G(S') \cap N_G(S)| = |N_{G-G'}(S')| + |S| < |S'| + |S|$$
$$= |S' \cup S|$$

which contradicts our assumption. Again, by induction, $G - G'$ contains a matching M_2 of $X - S$. The union of the two matchings $M_1 \cup M_2$ is a matching of X.

Corollary: m-regular bipartite graph can be edge colored by m colors.
Proof. Consider a bipartite graph $G(X, Y, E)$ with $\delta(v) = m$, $\forall\, v \in G$. Define

$$E(S) \coloneqq \{e | e \text{ is incident to a vertex } v \in S\},$$

we have

$$E(S) \subseteq E\big(N(S)\big), \qquad \forall\, S \subseteq X.$$

Therefore

$$|E(S)| \leq \big|E\big(N(S)\big)\big|$$

Because $|E(S)| = m|S|$, and $\big|E\big(N(S)\big)\big| = m|N(S)|$, we have

$$m|S| \leq m|N(S)|$$

$$\Longrightarrow \quad |N(S)| \geq |S|$$

which satisfies Hall's condition, and thus a matching M exists in G. Since the bipartite graph

$$G' = G - M$$

is also regular with $\delta(v) = m - 1,\ \forall\, v \in G'$. Therefore we can use the above strategy repeatedly until $\delta(v) = 0$. As a result, we get a total of m matching in G, each corresponds to a color, hence G can be edge colored by m colors.

(Prepared by Miao Zhang, May 29, 2011)