Nonblocking Conditions for Flex-grid OXC-Clos Networks

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***Abstract*—The emerging high-capacity optical networks makes it urgent to design large-scale flexible mesh optical cross-connects (OXCs). Though Clos network is the theory for building scalable and cost-effective switching fabrics, the nonblocking conditions of flex-grid optical Clos networks without wavelength conversion remain unknown. This paper studies the nonblocking conditions for the flex-grid OXC-Clos network, which is constructed from a number of small-size standard OXCs. We first show that a strictly nonblocking (SNB) OXC-Clos network will incur a high cost, as small-granularity lightpaths may abuse central modules, rendering them unavailable for large-granularity requests due to frequency conflicts. We thus propose a granularity differential routing (GDR) strategy, the idea of which is to restrict the set of CMs that can be used by the lightpaths of each granularity. Under the GDR strategy, we investigate two system models, granularity-port binding and unbinding models, and prove the wide-sense nonblocking (WSNB) conditions for OXC-Clos network. We show that the cost of WSNB network is remarkably smaller than that of SNB network, and find that the second model can lead to more flexible network-bandwidth utilization than the first model only at a small cost of switching fabrics.**

***Index Terms*—Optical cross-connect, flex-grid optical network, wide-sense nonblocking, strictly nonblocking.**

I. Introduction

In recent years, the surge in Internet traffic caused by high-performance computing services and multimedia services is driving the continuous growth of optical network capacity. On one hand, the bandwidth of one-fiber links will be exhausted and the use of multiple fibers on optical links has been put on the agenda. As Ref. [1] points out, optical fiber deployment is growing at an annual rate of 15%. On the other hand, the data rates of optical signals launched into the optical links climb up from 10Gb/s to 100Gb/s, and will soon reach 400Gb/s and above. It is very necessary to maximize spectral efficiency according to the data rate and the transmission distance of each demand [2]. In this context, the traditional fixed-grid optical networks that divide the optical spectrum into fixed wavelength grids will no longer work well, and flex-grid optical networks are emerging to provide high spectral efficiency. Accordingly, the design of network device should adapt to the new changes.

Meanwhile, optical cross-connect (OXC) has become the key component of switching nodes in optical networks. As Fig. 1 shows, a classical $N \times N$ OXC includes N $1 \times N$ wavelength selective switches (WSSes) at the input side and N $N \times 1$ WSSes at the output side, with an $N^2 \times N^2$ shuffle interconnection network in between. The OXC is a flexible optical switching fabric. If there is a common spectrum interval free on an input and an output, the OXC can establish a lightpath in a strictly nonblocking (SNB) manner. Fig. 1 displays three lightpaths occupying different spectral bandwidths. However, the scalability of classical OXC is restricted by the WSS, the port count of which is currently limited to 49 [3]. The classical OXC cannot meet the application requirement of future optical networks, which will require optical nodes with hundreds of ports due to the use of multiple fibers on each optical link. Thus, it is highly desired to enhance the scalability of OXCs.

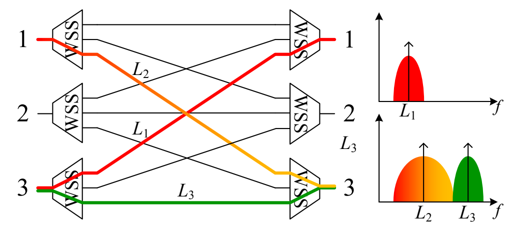

Fig. 1. A standard 3×3 OXC, where L_i stands for lightpath i and different colors represent different optical frequencies.

Clos network is the theory to construct a scalable and cost-effective switching fabric. In the past decades, different electrical/optical Clos switches [4]–[9] and the related nonblocking conditions have been studied. However, only a few endeavors have been made to apply this theory to construct scalable flex-grid OXCs, called *flex-grid OXC-Clos network* in this paper, whose nonblocking conditions are still unknown.

This paper studies the nonblocking conditions for flex-grid OXC-Clos networks without wavelength converters (WCs). In particular, we focus on the ability of flex-grid OXC-Clos networks to establish new lightpaths without any reconfiguration, which is of great significance for industrial applications. Our goal is to remarkably reduce the number of central modules (CMs) needed by a nonblocking flex-grid OXC-Clos network.

We devise a granularity differential routing (GDR) strategy, motivated by the derivation of the SNB condition for flex-grid OXC-Clos networks. We demonstrate that the small-granularity lightpaths in an SNB OXC-Clos network may occupy too many CMs, which then become unavailable for future large-granularity requests due to spectral conflicts. Thus, a lot of CMs are needed by an SNB flex-grid OXC-Clos network. The key idea of our GDR strategy is to restrict the set of CMs that can be used by the lightpaths of each granularity, thereby leaving more CMs for large-granularity requests. Under the GDR strategy, we prove the wide-sense nonblocking (WSNB) conditions for flex-grid OXC-Clos network. We show that the number of CMs needed by a WSNB OXC-Clos network is remarkably smaller than that by an SNB network.

This paper studies two system models under the flex-grid scenario. The first one, named granularity-port binding (GPB) model, stems from a practical application scenario, where each port of the switch only carries the lightpaths with the same granularity. The second one, called granularity-port unbinding (GPuB) model, allows the lightpaths of different granularities to coexist in a port. Compared to the GPB model, the GPuB model introduces more flexibility in port utilization. We show that the sufficient WSNB condition for the GPB model is the necessary and sufficient WSNB condition for the GPuB model, indicating that flexibly deploying lightpaths among the ports only leads to a slightly high cost.

The rest of this paper is organized as follows. Section II introduces the OXC-Clos network and two system models, and analyzes the cost of SNB networks, according to which we propose the GDR strategy and derive the WSNB conditions under the GPB and GPuB models in Section III and Section IV, respectively. The related works are reviewed in Section V. Section VI concludes this paper.

II. PRELIMINARY

As Fig. 2 displays, an $N \times N$ symmetric flex-grid OXC-Clos network, denoted by $\mathcal{C}(n, r, m)$, includes r $n \times m$ input modules (IMs) at the input stage, m $r \times r$ central modules (CMs) at the central stage, and r $m \times n$ output modules (OMs) at the output stage, where each module is a standard OXC module and $N = n \times r$. The two OXC modules at the adjacent stages are connected by a single fiber. From top to bottom, we number the IMs by $1, \cdots, \alpha, \cdots, r$, the CMs by $1, \cdots, \gamma, \cdots, m$, and the OMs by $1, \cdots, \beta, \cdots, r$. Similarly, we number the inputs of each IM as $1, \cdots, a, \cdots, n$, and the outputs of each OM as $1, \cdots, b, \cdots, n$.

In a flex-grid optical network, the spectrum of each port is divided into W basic slots (bSlots), the width of which is 12.5 GHz [10]. In Fig. 2, a bSlot is represented by a slot. We index W bSlots by $1, \cdots, w, \cdots, W$. A set of ω adjacent bSlots indexed by $w, w+1, \cdots, w+\omega-1$ defines an ω-granularity (ω-g) frequency slot, which is denoted by $\mathbf{\Lambda}_w^\omega$ in this paper, where w and ω are two positive integers and $w+\omega-1 \leq W$. The "flex-grid" means that the frequency slots with different granularities can coexist in the network.

This paper considers the communication mode, in which the network sets up an ω-g lightpath from an input to an output if there is a frequency slot $\mathbf{\Lambda}_w^\omega$ free on them. We denote an ω-g lightpath from input a of IM α to output b of OM β as $L\left(\alpha, a, \beta, b, \mathbf{\Lambda}_w^\omega\right)$. Similarly, a lightpath request is denoted by $R\left(\alpha, a, \beta, b, \mathbf{\Lambda}_w^\omega\right)$. For example, L_1 in Fig. 2 is a 2-g lightpath, denoted by $L\left(\alpha, 1, \beta, 2, \mathbf{\Lambda}_1^2\right)$, and L_2 and L_3 are two 1-g

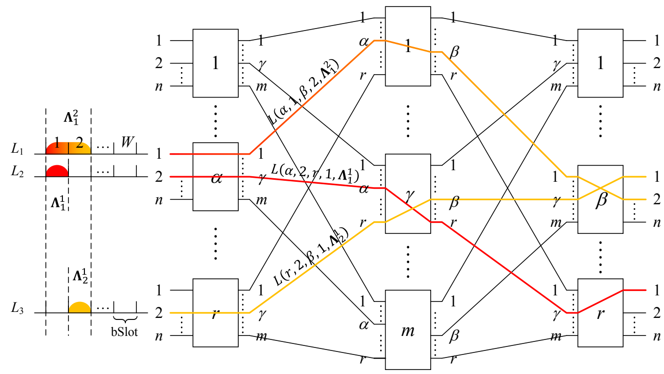

Fig. 2. A flex-grid optical Clos Network $\mathcal{C}(n, r, m)$ with 3 lightpaths.

lightpaths, denoted by $L\left(\alpha, 2, r, 1, \mathbf{\Lambda}_1^1\right)$ and $L\left(r, 2, \beta, 1, \mathbf{\Lambda}_2^1\right)$. Also, the spectrum interval used by a lightpath is represented by a colored box covering several slots in the figure. As an instance, the spectrum interval of L_1 includes bSlots 1 and 2.

The OXC-Clos network has two routing constraints. That is, two lightpaths cannot use the same CM, if

C1. they share the same IM and occupy the same bSlots, or

C2. they share the same OM and occupy the same bSlots.

The example for constraint C1 is L_1 and L_2 in Fig. 2 and that for C2 is L_1 and L_3 in Fig. 2.

Definition 1. *An OXC is SNB if a lightpath can always be set up between an input and an output without rearranging the paths of the existing lightpaths when the input and output have the same idle spectrum interval.*

Definition 2. *An OXC is WSNB if a routing strategy exists for setting lightpaths in such a way that a lightpath can always be set up between an input and an output without rearranging the paths of the existing lightpaths when the input and output have the same idle spectrum interval.*

It is trivial to show that one CM is enough for $\mathcal{C}$ with $n = 1$ to be SNB. In this case, $\mathcal{C}$ reduces to a classical OXC. Thus, this paper only considers the case where $n > 1$.

A. System Models

To facilitate our study, we assume that there are K types of lightpaths (or frequency slots) in the network and

A1. there are $W = 2^{K-1}$ bSlots,

A2. the k-th type of lightpath occupies 2^{k-1} adjacent bSlots, where $k = 1, 2, \cdots, K$.

Note that the idea of our approach in this paper is not limited to the granularity pattern presented in A2 and can be applied to other kinds of granularity patterns.

Also, we consider two models in this paper.

M1. **Granularity-Port Binding (GPB) model**: Once a port is used by a type of lightpaths, this port can only carry this type of lightpaths until it becomes completely free, which means all the lightpaths on this port are torn down.

Fig. 3(a) plots an $N \times N$ OXC, each port of which carries 8 bSlots. Input 1 only carries 2-g lightpaths once it is occupied

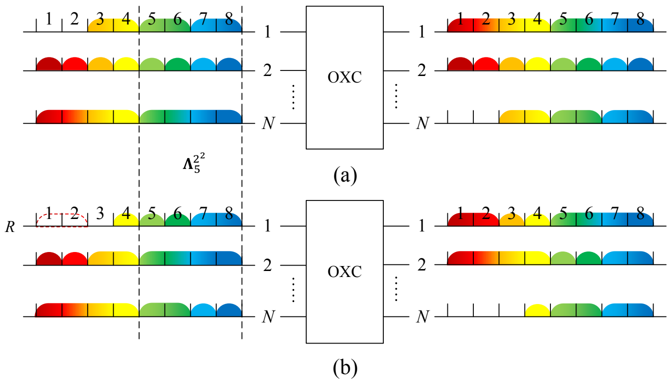

Fig. 3. Examples of (a) GPB model and (b) GPuB model, where $K = 4$ and request R is represented by a dotted box.

by a 2-g lightpath. If input 1 becomes completely idle after all the 2-g lightpaths are torn down, it can be rebound by other types of lightpaths.

The GPB model stems from practical applications. There is one type of most cost-effective transceiver, when building an optical network. It is a common practice in real networks to adopt one modulation format with one granularity, which can simplify network management and maintenance. Hence, the single-fiber network typically has only one granularity of optical signals. When the network is upgraded by installing more fibers, a different type of transceiver with another granularity may likely be added to match the newly installed fibers. In this case, the flex-grid optical network will be possibly deployed according to the GPB model in the near future.

To avoid bandwidth fragmentation and maximize bandwidth utilization under the GPB model, this paper makes the following restriction:

A3. a 2^k-g lightpath is carried by a 2^k-g frequency slot, and the i-th 2^k-g frequency slot is defined by the set

$$\mathbf{\Lambda}_{2^k(i-1)+1}^{2^k} = \left\{2^k(i-1)+1, \cdots, 2^k i\right\}.$$

For example, $\mathbf{\Lambda}_5^{2^2} = \{5, 6, 7, 8\}$ in Fig. 3(a) defines the second 2^2-g frequency slot, while bSlots 3 through 6 do not form a 2^2-g frequency slot. The violation of A3 may lead to bandwidth fragmentation. Consider the following case. If input N in Fig. 3(a) carries a 2^2-g lightpath using bSlots 3 through 6, bSlots 1, 2, 7, and 8 will be wasted under the GPB model until this input becomes completely idle.

A3 implies that a 2^k-g frequency slot can accommodate up to 2^{k-i} 2^i-g lightpaths or be occupied by a 2^j-g lightpath, where $i = 0, 1, \cdots, k-1$, and $j = k, k+1, \cdots, K-1$.

We notice that the flex-grid networks have just been built for less than 10 years. As traffic increases, the optical network will undergo multiple upgrades. In this case, the GPB model may lead to an inflexible utilization of network-link capacity. For example, although input 1 in Fig. 3(a) has two free bSlots, it cannot offer bandwidth to a 1-g request due to granularity-port binding. Allowing multiple granularities coexisting in the same fiber would be a better choice. We thus slightly relax the constraint imposed by granularity-port binding and explore a more flexible model as follows.

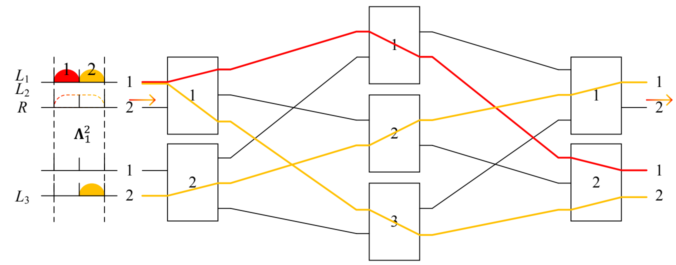

Fig. 4. Flex-grid OXC-Clos Network $\mathcal{C}(2, 2, 3)$ is not SNB, where request R is represented by a dotted box.

M2. **Granularity-Port unBinding (GPuB) model**: Each port can carry different types of lightpaths.

The "*slightly relax*" means we still consider A3 in the GPuB model. Fig. 3(b) illustrates the GPuB model, where input 1 has 3 idle bSlots. As A3 specifies, if a 2-g request from input 1 to output N arrives, input 1 will allocate bSlots 1 and 2 to this request, instead of bSlots 2 and 3.

B. Cost of SNB Flex-grid Clos Network

The SNB condition for traditional Clos networks, i.e., $m \geq 2n-1$ [4], cannot be applied to flex-grid OXC-Clos Networks. Fig. 4 is an OXC-Clos network $\mathcal{C}(2, 2, 3)$, where there are three existing lightpaths and $m \geq 2n-1 = 3$. However, as Fig. 4 plots, $R(1, 2, 1, 2, \mathbf{\Lambda}_1^2)$ is blocked, since lightpaths L_1 and L_2 share the same IM with R and use CMs 1 and 3 while lightpath L_3 shares the same OM with R and passes through CM 2. No CM is available for R due to frequency conflicts. In fact, the number of CMs needed by an SNB flexible OXC-Clos is very large, as we show in the following theorem.

Theorem 1. *When K types of lightpaths coexist, $\mathcal{C}(n, r, m)$ is SNB iff*

$$m \geq 2^K(n-1) + 1. \tag{1}$$

Proof. Suppose there is a request $R\left(\alpha, a, \beta, b, \mathbf{\Lambda}_1^{2^{K-1}}\right)$ from input a of IM α to output b of OM β. Consider the worst case, where frequency slot $\mathbf{\Lambda}_1^{2^{K-1}}$ is busy in carrying 2^{K-1} 1-g lightpaths on all other inputs of IM α and all other outputs of OM β.

Let $\boldsymbol{S}_\alpha$ and $\boldsymbol{S}_\beta$ be the set of CMs used by the 1-g lightpaths from IM α and that used by the 1-g lightpaths ahead to OM β, respectively. Clearly,

$$|\boldsymbol{S}_\alpha| \leq 2^{K-1}(n-1),$$

and

$$|\boldsymbol{S}_\beta| \leq 2^{K-1}(n-1),$$

where the equalities hold if each lightpath uses a separate CM. Also,

$$|\boldsymbol{S}_\alpha \cup \boldsymbol{S}_\beta| \leq |\boldsymbol{S}_\alpha| + |\boldsymbol{S}_\beta| = 2^K(n-1),$$

where the inequality holds with equality when $\boldsymbol{S}_\alpha \cap \boldsymbol{S}_\beta = \emptyset$.

According to routing constraints C1 and C2, request R can be satisfied only when there is at least one more CM that is not used by the lightpaths that originate from IM α or the lightpaths that go ahead to OM β. We thus need

$$m = 2^K(n-1)+1$$

CMs to accommodate R. $\square$

Clearly, (1) immediately reduces to the SNB condition of traditional Clos network when $K=1$, i.e., there is only one type of lightpaths in the network.

Theorem 1 shows that the number of CMs required by an SNB flex-grid OXC-Clos network is large, which is attributed to the fact that small-granularity lightpaths (e.g., 1-g lightpaths) may abuse the CMs such that a large number of CMs will not be available for future large-granularity requests. As Fig. 4 plots, the two 1-g lightpaths L_1 and L_2 at input 1 of IM α use two different CMs, though they can share the same CM since they use different bSlots. This motivates us to devise a routing strategy, which restricts the set of CMs occupied by small-granularity lightpaths.

III. WSNB Condition under GPB Model

In this section, we will develop a routing strategy under the GPB model to restrict the set of CMs occupied by small-granularity lightpaths, such that the number of CMs needed by a nonblocking OXC-Clos network can be reduced. The key problem is to figure out the set of CMs that each type of lightpaths can employ for routing. We solve this problem in an inductive manner. Specifically, we first find the sets of CMs that can be used by 1-g lightpaths and 2-g lightpaths in section III-A and III-B, from which section III-C then proposes the GDR strategy and proves the WSNB condition.

A. CMs that 1-g Lightpaths can Use

We first determine the minimal number of CMs required by $\mathcal{C}$ to route 1-g lightpaths without reconfiguration when all K types of lightpaths coexist. With this information, we can then specify the set of CMs, via which the 1-g lightpaths should be routed.

Definition 3. *In the case where $\mathcal{C}$ needs to support K types of lightpaths, $\mathcal{C}$ is SNB for 2^k-g lightpaths if it can always satisfy a 2^k-g request $R\left(\alpha,a,\beta,b,\mathbf{\Lambda}_w^{2^k}\right)$ without reconfiguration as long as frequency slot $\mathbf{\Lambda}_w^{2^k}$ is available on both input a of IM α and output b of OM β, where w is the first bSlot of a 2^k-g frequency slot and $k=0,1,\cdots,K-1$.*

Lemma 1. *$\mathcal{C}$ is SNB for 1-g lightpaths under the GPB model iff $m \geq 2n-1$.*

Proof. Suppose there is a request $R\left(\alpha,a,\beta,b,\mathbf{\Lambda}_w^1\right)$. Consider the case, where

(a) All other $n-1$ inputs of IM α are busy carrying $n-1$ lightpaths, each of which occupies $\mathbf{\Lambda}_w^1$;

(b) All other $n-1$ outputs of OM β are busy carrying $n-1$ lightpaths, each of which occupies $\mathbf{\Lambda}_w^1$.

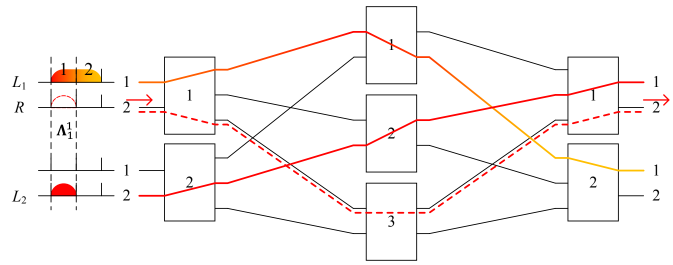

Fig. 5. $\mathcal{C}(2,2,3)$ is SNB for 1-g lightpaths though it carries two types of lightpaths.

The granularity of the lightpaths carried by each port is the same and could be any of $1,2,\cdots$, or 2^{K-1}.

Let $\boldsymbol{S}_\alpha$ be the set of CMs used by the lightpaths that occupy $\mathbf{\Lambda}_w^1$ and originate from IM α, and $\boldsymbol{S}_\beta$ be the set of CMs used by the lightpaths that occupy $\mathbf{\Lambda}_w^1$ and go ahead to OM β. Clearly, $|\boldsymbol{S}_\alpha| = |\boldsymbol{S}_\beta| = n-1$. Also,

$$|\boldsymbol{S}_\alpha \cup \boldsymbol{S}_\beta| \leq |\boldsymbol{S}_\alpha| + |\boldsymbol{S}_\beta| = 2(n-1),$$

where the inequality holds with equality when $\boldsymbol{S}_\alpha \cap \boldsymbol{S}_\beta = \emptyset$.

According to constraints C1 and C2, R can be satisfied only if there is at least one CM that is not used by the lightpaths mentioned in (a) and (b). We thus need

$$m = 2n-1$$

CMs to accommodate R, which proves this lemma. $\square$

Lemma 1 is illustrated in Fig. 5, where 3 CMs are enough to satisfy a 1-g request R without any reconfiguration, though a 2-g lightpath L_1 and a 1-g lightpath L_2 are in the network.

B. CMs that 2-g Lightpaths can Use

Lemma 1 allows restricting the routing of all 1-g lightpaths to a set of $2n-1$ CMs, such that more CMs can be left for the requests with larger granularity. We are now ready to check the minimum number of CMs required to route 1-g lightpaths and 2-g lightpahts without any reconfiguration when there is a routing restriction for 1-g lightpaths as follows.

Routing Strategy for 1-g lightpaths:

1) Specify a fixed set of $2n-1$ CMs, denoted by $\boldsymbol{M}_0$, via which all 2^0-g lightpaths can only be routed;
2) Lightpaths with granularity larger than 2^0 can be routed via the set of all CMs, denoted by $\boldsymbol{M}$.

Herein, we specify $\boldsymbol{M}_0 = \{1,2,\cdots,2n-1\}$.

Definition 4. *In the case where $\mathcal{C}$ needs to support K types of lightpaths, $\mathcal{C}$ is WSNB for 2^k lightpaths if it can always satisfy a 2^k-g request $R\left(\alpha,a,\beta,b,\mathbf{\Lambda}_w^{2^k}\right)$ without reconfiguration under a routing strategy as long as frequency slot $\mathbf{\Lambda}_w^{2^k}$ is free on both input a of IM α and output b of OM β, where w is the first bSlot of a 2^k-g frequency slot and $k=0,1,\cdots,K-1$.*

Before finding the minimal number of CMs required to route 1-g and 2-g lightpaths without reconfiguration when $\mathcal{C}$ has to support K types of lightpaths, we need the following lemma.

Lemma 2. *$f(x) = x - \lceil x/2 + a \rceil$ is non-decreasing, where $x = 1, 2, \cdots$ and a is an integer constant.*

Lemma 3. *Under Routing Strategy for 1-g lightpaths, $\mathcal{C}$ is WSNB for both 1-g and 2-g lightpaths under the GPB model iff $m \geq 3n - 2$.*

Proof. Consider a request $R(\alpha, a, \beta, b, \mathbf{\Lambda}_w^2)$ that sees the following situation:

(a) p_α inputs of IM α and p_β outputs of OM β are busy carrying 1-g lightpaths via $\mathbf{\Lambda}_w^2$, and

(b) $n - p_\alpha - 1$ inputs of IM α and $n - p_\beta - 1$ outputs of OM β are busy carrying the last $K - 1$ types of lightpaths via $\mathbf{\Lambda}_w^2$, and each port only carries one type of lightpaths,

where $p_\alpha, p_\beta = 0, 1, \cdots, n - 1$.

Let $\boldsymbol{B}_0$ be the set of CMs used by the 1-g lightpaths that are carried by IM α or OM β via $\mathbf{\Lambda}_w^2$. According to lemma 1, all 1-g lightpaths can be routed via the CMs in $\boldsymbol{M}_0$, which means

$$|\boldsymbol{B}_0| \leq |\boldsymbol{M}_0| = 2n - 1. \tag{2}$$

Let $\boldsymbol{S}_\alpha$ be the set of CMs used by the last $K - 1$ types of lightpaths that occupy $\mathbf{\Lambda}_w^2$ and originate from IM α, and $\boldsymbol{S}_\beta$ be the set of CMs used by the last $K - 1$ types of lightpaths that occupy $\mathbf{\Lambda}_w^2$ and go ahead to OM β. Note that a lightpath with granularity larger than 1 cannot share the same CM with another one that also uses the bSlots in $\mathbf{\Lambda}_w^2$, if both of them originate from IM α or go ahead to OM β. It follows that

$$|\boldsymbol{S}_\alpha| = n - p_\alpha - 1,$$
$$|\boldsymbol{S}_\beta| = n - p_\beta - 1,$$

and thus

$$|\boldsymbol{S}_\alpha \cup \boldsymbol{S}_\beta| \leq |\boldsymbol{S}_\alpha| + |\boldsymbol{S}_\beta| = 2(n-1) - (p_\alpha + p_\beta). \tag{3}$$

where the inequality holds with equality when $S_\alpha \cap S_\beta = \emptyset$.

$p_\alpha + p_\beta$ in (3) can be determined as follows. Any two 1-g lightpaths carried by p_α inputs of IM α or p_β outputs of OM β via $\mathbf{\Lambda}_w^2$ can share the same CM, as long as they do not use the same bSlot. This implies that $|\boldsymbol{B}_0|$ must be less or equal to the number of these lightpaths. Furthermore, the total number of bSlots in $\mathbf{\Lambda}_w^2$ on p_α inputs of IM α and p_β outputs of OM β is $2(p_\alpha + p_\beta)$, which can carry up to $2(p_\alpha + p_\beta)$ 1-g lightpaths. Thus, we have $|\boldsymbol{B}_0| \leq 2(p_\alpha + p_\beta)$ or

$$p_\alpha + p_\beta \geq \left\lceil \frac{|\boldsymbol{B}_0|}{2} \right\rceil. \tag{4}$$

It follows from (3) through (4) that the set of CMs that are not available for request R satisfies

$$\begin{aligned}
|\boldsymbol{B}_0 \cup \boldsymbol{S}_\alpha \cup \boldsymbol{S}_\beta| &\leq |\boldsymbol{B}_0| + |\boldsymbol{S}_\alpha \cup \boldsymbol{S}_\beta| \\
&\leq |\boldsymbol{B}_0| + 2(n-1) - (p_\alpha + p_\beta) \\
&\leq |\boldsymbol{B}_0| + 2(n-1) - \left\lceil \frac{|\boldsymbol{B}_0|}{2} \right\rceil \\
&\leq |\boldsymbol{M}_0| + 2n - 2 - \left\lceil \frac{|\boldsymbol{M}_0|}{2} \right\rceil \\
&= 2n - 1 + 2n - 2 - n \\
&= 3n - 3,
\end{aligned} \tag{5}$$

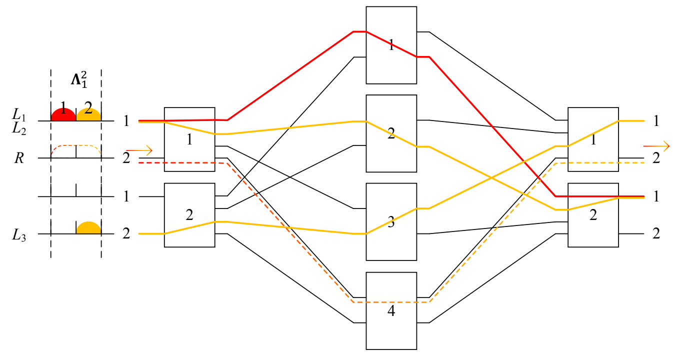

Fig. 6. $\mathcal{C}(2, 2, 4)$ is WSNB for 2^0-g and 2^1-g lightpaths, where $\boldsymbol{M}_0 = \{1, 2, 3\}$ and $\boldsymbol{M}_1 = \{1, 2, 3, 4\}$.

where we use lemma 2 and (2) for the fourth inequality. For an arbitrary $n \geq 2$, the inequality of (5) holds for equality if R sees the following situation when it arrives:

(i) In IM α, each of the $n - 1$ inputs carries 2 2^0-g lightpath via $\mathbf{\Lambda}_w^2$, and

(ii) In OM β, one output carries 1 2^0-g lightpath via $\mathbf{\Lambda}_w^2$, and each of the other $n - 2$ outputs carries 1 2^1-g lightpath via $\mathbf{\Lambda}_w^2$.

All the lightpaths in (i) and (ii) are different and use different CMs. In particular, $2n - 1$ 2^0-g lightpaths are routed via CMs $1, 2, \cdots, 2n - 1$, and $n - 2$ 2^1-g lightpaths are routed via CMs $2n, 2n+1, \cdots, 3n-3$, which conforms with the GDR strategy.

We thus need $m \geq 3n - 2$ CMs to satisfy R. ☐

Lemma 3 implies the OXC-Clos network is WSNB for both 1-g and 2-g lightpaths, under the routing strategy as follows:

1) Specify a CM set $\boldsymbol{M}_0 = \{1, 2, \cdots, 2n - 1\}$ for all 2^0-g lightpaths, via which 2^0-g lightpaths can only be routed;
2) Specify a CM set $\boldsymbol{M}_1 = \{1, 2, \cdots, 3n - 2\}$ for all 2^1-g lightpaths, via which 2^1-g lightpaths can only be routed;
3) Lightpaths with granularity larger than 2^1 can be routed via the set of all CMs, denoted by $\boldsymbol{M}$.

Fig. 6 illustrates the routing strategy for 2^0-g and 2^1-g lightpaths in $\mathcal{C}(2, 2, 4)$, where the 2^0-g lightpaths L_1, L_2, and L_3 are routed via CMs 1, 2, 3, and the 2^1-g lightpaths can employ all the CMs.

C. GDR strategy and WSNB condition

In this part, we generalize lemmas 1 and 3 to propose the GDR strategy for K types of lightpaths, and prove the WSNB condition for flex-grid OXC-Clos networks.

The GDR Strategy in general case is as follows:

GDR Strategy

1) Specify a set of CMs

$$\boldsymbol{M}_i = \{1, 2, \cdots, 2n - 1 + i(n - 1)\}$$

for 2^i-g lightpaths, via which all the 2^i-g lightpaths can only be routed, where $i = 0, 1, \cdots, K - 2$;

2) 2^{K-1}-g lightpaths can be routed via the set of all CMs, denoted by $\boldsymbol{M}$. Clearly, $\boldsymbol{M}_{K-1} = \boldsymbol{M}$.

It is easy to know that the GDR strategy is quite simple and does not increase the operation complexity.

We will derive the minimal value of $|\boldsymbol{M}|$ and prove that $\mathcal{C}$ with the GDR strategy is WSNB when it needs to support K types of lightpaths. Before that, we need the following lemma.

Lemma 4. *Consider J sets $\boldsymbol{F}_0, \boldsymbol{F}_1, \cdots, \boldsymbol{F}_{J-1}$ and $\widehat{\boldsymbol{F}}_j \triangleq \bigcup_{i=0}^{j} \boldsymbol{F}_i$, where $j = 0, 1, \cdots, J-1$. The following inequality*

$$\sum_{j=0}^{J-1} \left(2^j |\boldsymbol{F}_j|\right) \geq 2^{J-1} \left|\widehat{\boldsymbol{F}}_{J-1}\right| - \sum_{j=0}^{J-2} \left(2^j \left|\widehat{\boldsymbol{F}}_j\right|\right)$$

always holds, and the inequality is satisfied with equality when $\boldsymbol{F}_0, \boldsymbol{F}_1, \cdots, \boldsymbol{F}_{J-1}$ are mutually disjoint.

Proof. Since

$$\left|\widehat{\boldsymbol{F}}_j\right| = \left|\widehat{\boldsymbol{F}}_{j-1}\right| + |\boldsymbol{F}_j| - \left|\widehat{\boldsymbol{F}}_{j-1} \cap \boldsymbol{F}_j\right|,$$

$\forall j = 1, 2, \cdots, J-1$, we have

$$\begin{aligned}
&\sum_{j=0}^{J-1} \left(2^j |\boldsymbol{F}_j|\right) \\
=& |\boldsymbol{F}_0| + 2|\boldsymbol{F}_1| + \cdots + 2^j |\boldsymbol{F}_j| + \cdots + 2^{J-1} |\boldsymbol{F}_{J-1}| \\
=& \left|\widehat{\boldsymbol{F}}_0\right| + \cdots + 2^j \left(\left|\widehat{\boldsymbol{F}}_j\right| - \left|\widehat{\boldsymbol{F}}_{j-1}\right| + \left|\widehat{\boldsymbol{F}}_{j-1} \cap \boldsymbol{F}_j\right|\right) + \cdots \\
&+ 2^{J-1} \left(\left|\widehat{\boldsymbol{F}}_{J-1}\right| - \left|\widehat{\boldsymbol{F}}_{J-2}\right| + \left|\widehat{\boldsymbol{F}}_{J-2} \cap \boldsymbol{F}_{J-1}\right|\right) \\
=& -\left|\widehat{\boldsymbol{F}}_0\right| - 2\left|\widehat{\boldsymbol{F}}_1\right| - \cdots - 2^{J-2}\left|\widehat{\boldsymbol{F}}_{J-2}\right| + 2^{J-1}\left|\widehat{\boldsymbol{F}}_{J-1}\right| \\
&+ \left(2\left|\widehat{\boldsymbol{F}}_0 \cap \boldsymbol{F}_1\right| + \cdots + 2^{J-1}\left|\widehat{\boldsymbol{F}}_{J-2} \cap \boldsymbol{F}_{J-1}\right|\right) \\
\geq& -\left|\widehat{\boldsymbol{F}}_0\right| - 2\left|\widehat{\boldsymbol{F}}_1\right| - \cdots - 2^{J-2}\left|\widehat{\boldsymbol{F}}_{J-2}\right| + 2^{J-1}\left|\widehat{\boldsymbol{F}}_{J-1}\right| \\
=& 2^{J-1}\left|\widehat{\boldsymbol{F}}_{J-1}\right| - \sum_{j=0}^{J-2}\left(2^j \left|\widehat{\boldsymbol{F}}_j\right|\right).
\end{aligned}$$

It is clear that the equality holds when $\widehat{\boldsymbol{F}}_{j-1} \cap \boldsymbol{F}_j = \emptyset$, that is, $\boldsymbol{F}_0, \boldsymbol{F}_1, \cdots, \boldsymbol{F}_{J-1}$ are mutually disjoint. □

Under the GPB model, we only obtain the sufficient condition in general case for WSNB $\mathcal{C}$, which is required to support K types of lightpaths.

Theorem 2. *$\mathcal{C}$ is WSNB for the 2^0-g, 2^1-g,$\cdots$, and 2^k-g lightpaths under the GPB model, if*

$$m \geq 2n - 1 + k(n-1), \tag{6}$$

where $k = 0, 1, \cdots, K-1$.

Proof. Lemmas 1 and 3 show this theorem holds for $k = 0$ and 1. We prove that if this theorem is true for $k = i$, where $i = 2, 3, \cdots, K-2$, it also holds for $k = i+1$.

Consider a 2^{i+1}-g request $R\left(\alpha, a, \beta, b, \boldsymbol{\Lambda}_w^{2^{i+1}}\right)$ that sees the following situation:

(a) p_α^j inputs of IM α and p_β^j outputs of OM β are busy in carrying 2^j-g lightpaths via $\boldsymbol{\Lambda}_w^{2^{i+1}}$, and
(b) $n - \sum_{j=0}^{i} p_\alpha^j - 1$ inputs of IM α and $n - \sum_{j=0}^{i} p_\beta^j - 1$ outputs of OM β are busy in carrying the last $K-i-1$ types of lightpaths via $\boldsymbol{\Lambda}_w^{2^{i+1}}$, and each port only carries one type of lightpaths,

where $p_\alpha^j, p_\beta^j = 0, 1, \cdots, n-1$ and $j = 0, 1, \cdots, i$. The total number of ports busy in $\boldsymbol{\Lambda}_w^{2^{i+1}}$ at IM α and OM β is $2n-2$.

Let $\boldsymbol{B}_j$ be the set of CMs used by the 2^j-g lightpaths that are carried by IM α or OM β via $\boldsymbol{\Lambda}_w^{2^{i+1}}$. Define $\widehat{\boldsymbol{B}}_j = \bigcup_{l=0}^{j} \boldsymbol{B}_l$. According to the GDR strategy and the induction hypothesis, all lightpaths in $\widehat{\boldsymbol{B}}_j$ can be routed via the CMs in $\boldsymbol{M}_j$ under the GDR strategy. Thus, there is

$$\left|\widehat{\boldsymbol{B}}_j\right| \leq |\boldsymbol{M}_j|. \tag{7}$$

Note that, for some combinations of n and i, the equality of (7) may be always unachievable in the GPB model. For example, consider the case where $i = 2n-1$. As a port of $\mathcal{C}$ can only carry one type of lightpaths, R cannot see at least one type of lightpaths at the $2n-2$ ports busy in $\boldsymbol{\Lambda}_w^{2^{i+1}}$ at IM α and OM β. In this case, if the 2^j-g lightpath does not appear, there is

$$\left|\widehat{\boldsymbol{B}}_j\right| = \left|\widehat{\boldsymbol{B}}_{j-1}\right| \leq |\boldsymbol{M}_{j-1}| < |\boldsymbol{M}_j|,$$

where $j = 1, 2, \cdots, i$.

Let $\boldsymbol{S}_\alpha$ be the set of CMs used by the last $K-i-1$ types of lightpaths that occupy $\boldsymbol{\Lambda}_w^{2^{i+1}}$ and originate from IM α, and $\boldsymbol{S}_\beta$ be the set of CMs used by the last $K-i-1$ types of lightpaths that occupy $\boldsymbol{\Lambda}_w^{2^{i+1}}$ and go ahead to OM β. Note that a lightpath with granularity larger than 2^i cannot employ the same CM with another one that also uses the bSlots in $\boldsymbol{\Lambda}_w^{2^{i+1}}$, if they share IM α or OM β. It follows that

$$|\boldsymbol{S}_\alpha| = n - \sum_{j=0}^{i} p_\alpha^j - 1,$$

$$|\boldsymbol{S}_\beta| = n - \sum_{j=0}^{i} p_\beta^j - 1,$$

and thus

$$|\boldsymbol{S}_\alpha \cup \boldsymbol{S}_\beta| \leq |\boldsymbol{S}_\alpha| + |\boldsymbol{S}_\beta| = 2n - 2 - \sum_{j=0}^{i}\left(p_\alpha^j + p_\beta^j\right), \tag{8}$$

where the inequality holds with equality when $\boldsymbol{S}_\alpha \cap \boldsymbol{S}_\beta = \emptyset$.

$p_\alpha^j + p_\beta^j$ in (8) can be determined as follows. Any two 2^j-g lightpaths carried by p_α^j inputs of IM α or p_β^j outputs of OM β via $\boldsymbol{\Lambda}_w^{2^{i+1}}$ can share the same CM, as long as they do not use the same bSlot. This implies that $|\boldsymbol{B}_j|$ must be less or equal to the number of these lightpaths. Also, the total number of bSlots in $\boldsymbol{\Lambda}_w^{2^{i+1}}$ on p_α^j inputs of IM α and p_β^j outputs of OM β is $2^{i+1}\left(p_\alpha^j + p_\beta^j\right)$, which can carry up to $2^{i-j+1}\left(p_\alpha^j + p_\beta^j\right)$ 2^j-g lightpaths according to A3. Thus, we have

$$|\boldsymbol{B}_j| \leq 2^{i-j+1}\left(p_\alpha^j + p_\beta^j\right).$$

Multiplying both sides of the inequality by 2^j and summing over all js, we have

$$\sum_{j=0}^{i}\left(2^j |\boldsymbol{B}_j|\right) \leq 2^{i+1} \sum_{j=0}^{i}\left(p_\alpha^j + p_\beta^j\right).$$

TABLE I
CORRECTNESS OF THEOREM 1

n	K	# of CMs needed	$2n-1+(K-1)(n-1)$
2	4	6	6
11	4	51	51
4	6	21	22
11	6	71	71
2	10	11	12
5	10	45	45
7	10	66	67

It follows that

$$\begin{aligned}\sum_{j=0}^{i}\left(p_\alpha^j+p_\beta^j\right) &\geq \left\lceil \frac{\sum_{j=0}^{i}\left(2^j\left|\boldsymbol{B}_j\right|\right)}{2^{i+1}} \right\rceil \\ &\geq \left\lceil \frac{2^i\left|\widehat{\boldsymbol{B}}_i\right|-\sum_{j=0}^{i-1}\left(2^j\left|\widehat{\boldsymbol{B}}_j\right|\right)}{2^{i+1}} \right\rceil \\ &\geq \left\lceil \frac{2^i\left|\widehat{\boldsymbol{B}}_i\right|-\sum_{j=0}^{i-1}\left(2^j\left|\boldsymbol{M}_j\right|\right)}{2^{i+1}} \right\rceil, \end{aligned} \tag{9}$$

where we use lemma 4 for the second inequality, and (7) for the third inequality. It follows from (8) through (9) that the set of CMs that are not available for request R satisfies

$$\begin{aligned}&\left|\widehat{\boldsymbol{B}}_i\cup\boldsymbol{S}_\alpha\cup\boldsymbol{S}_\beta\right| \\ \leq&\left|\widehat{\boldsymbol{B}}_i\right|+\left|\boldsymbol{S}_\alpha\cup\boldsymbol{S}_\beta\right| \\ \leq&\left|\widehat{\boldsymbol{B}}_i\right|+2n-2-\sum_{j=0}^{i}\left(p_\alpha^j+p_\beta^j\right) \\ \leq&\left|\widehat{\boldsymbol{B}}_i\right|+2n-2-\left\lceil \frac{2^i\left|\widehat{\boldsymbol{B}}_i\right|-\sum_{j=0}^{i-1}\left(2^j\left|\boldsymbol{M}_j\right|\right)}{2^{i+1}} \right\rceil \\ \leq&\left|\boldsymbol{M}_i\right|+2n-2-\left\lceil \frac{2^i\left|\boldsymbol{M}_i\right|-\sum_{j=0}^{i-1}\left(2^j\left|\boldsymbol{M}_j\right|\right)}{2^{i+1}} \right\rceil \\ =&2n-1+i(n-1)+2n-2-n \\ =&2n-2+(i+1)(n-1), \end{aligned} \tag{10}$$

where we use lemma 2 and (7) for the fourth inequality. Since we use (7) in the derivation of (10), the equality of (10) may be always unachievable for some combinations of n and i. It is thus only sufficient to show R can be satisfied if there are

$$m \geq 2n-1+(i+1)(n-1)$$

CMs, which proves this theorem. □

To verify the correctness of theorem 2, we write a program to exhaustively enumerate all the cases for each pair of n and K to find the number of CMs needed by the worst case. Table I confirms that the number of CMs needed by each worst case is upper bounded by the sufficient condition (6). For example, when $n=2$ and $K=10$, the maximum number of required CMs is 11, which is less than

$$2n-1+(K-1)(n-1)=2\times2-1+9\times(2-1)=12.$$

TABLE II
A WORST CASE WHERE $n=2$ AND $K=10$.

i	0	1	2	3	4	5	6	7	8	9
$\left\lvert\widehat{\boldsymbol{B}}_i\right\rvert$	0	0	0	6	6	6	6	10	10	11
$\lvert\boldsymbol{M}_i\rvert$	3	4	5	6	7	8	9	10	11	12

Moreover, for $n=2$ and $K=10$, we check one of the worst cases, where a request $R\left(\alpha,1,\beta,1,\boldsymbol{\Lambda}_1^{2^9}\right)$ sees input 2 of IM α carries 6 2^3-g lightpaths and output 2 of OM β carries 4 2^7-g lightpaths via $\boldsymbol{\Lambda}_1^{2^9}$. The 6 2^3-g lightpaths employ CMs 1 through 6 and the 4 2^7-g lightpaths use CMs 7 through 10. As other 8 types of lightpaths do not appear at IM α and OM β, the equality of (7) may not hold, as Table II displays. For instance, $\left|\widehat{\boldsymbol{B}}_0\right|<\left|\boldsymbol{M}_0\right|$ since there is no 1-g lightpath. In this case, we need 11 CMs to satisfy the request R, meaning that $m=|M_9|\geq12$ is only the sufficient condition for WSNB.

IV. WSNB CONDITION UNDER GPUB MODEL

Different from the GPB model, the GPuB model allows each port to carry various types of lightpaths simultaneously. It is obvious that the GPuB model provides more flexibility for bandwidth utilization of optical networks. In this section, we study the WSNB condition for OXC-Clos networks under the GPuB model, following the GDR strategy. Our results show that the flexibility comes with the slight increase of the cost of OXC-Clos networks.

Similar to Section III, this part proves the WSNB condition in an inductive manner. Following the GDR strategy and the arguments used in Section III, we have the following lemmas.

Lemma 5. *$\mathcal{C}$ is SNB for 2^0-g lightpaths under the GPuB model iff $m\geq 2n-1$.*

Lemma 6. *$\mathcal{C}$ is WSNB for 2^0-g and 2^1-g lightpaths under the GPuB model iff $m\geq 3n-2$.*

The GPuB model allows different types of lightpaths to share the same port. When $i\geq 2n-1$, a 2^{i+1}-g request from IM α to OM β could see all $i+2$ types of lightpaths in its required frequency slot at IM α and OM β under the GPuB model, which is different from the situation under the GPB model as we show in theorem 2. Intuitively, the number of CMs occupied by the lightpaths that are carried by a given set of ports under the GPuB model would be larger than that under the GPB model. We will show that there always exists at least one case such that (7) holds with equality for any combination of n and i. Thus, the sufficient condition for WSNB OXC-Clos networks under the GPB model changes to the necessary and sufficient condition under the GPuB model, implying that the GPuB model increases the flexibility of bandwidth utilization only with a slightly increased cost.

Theorem 3. *$\mathcal{C}$ is WSNB for the 2^0-g, 2^1-g, $\cdots$, and 2^k-g lightpaths under the GPuB model, iff*

$$m\geq 2n-1+k(n-1), \tag{11}$$

where $k = 0, 1, \cdots, K-1$.

Proof. Lemmas 5 and 6 show this theorem holds for $k=0$ and 1. We prove that if this theorem is true for $k=i$, where $i=2,3,\cdots,K-2$, it also holds for $k=i+1$.

Consider the case where a newly arrived 2^{i+1}-g request $R\left(\alpha, a, \beta, b, \mathbf{\Lambda}_w^{2^{i+1}}\right)$ sees the following situation:

(a) p_α inputs of IM α and p_β outputs of OM β are busy in carrying the first $i+1$ types of lightpaths via $\mathbf{\Lambda}_w^{2^{i+1}}$, and

(b) $n-p_\alpha-1$ inputs of IM α and $n-p_\beta-1$ outputs of OM β are busy in carrying the last $K-i-1$ types of lightpaths via $\mathbf{\Lambda}_w^{2^{i+1}}$,

where $p_\alpha, p_\beta = 0, 1, \cdots, n-1$.

Let $\boldsymbol{B}_j$ be the set of CMs used by the 2^j-g lightpaths that are carried by IM α or OM β via $\mathbf{\Lambda}_w^{2^{i+1}}$. Define $\widehat{\boldsymbol{B}}_j = \bigcup_{l=0}^{j} \boldsymbol{B}_l$. According to the GDR strategy and the induction hypothesis, all lightpaths in $\widehat{\boldsymbol{B}}_j$ can be routed via the CMs in $\boldsymbol{M}_j$ under the GDR strategy. Thus, there is

$$\left|\widehat{\boldsymbol{B}}_j\right| \leq \left|\boldsymbol{M}_j\right|. \tag{12}$$

Let $\boldsymbol{S}_\alpha$ be the set of CMs used by the last $K-i-1$ types of lightpaths that occupy $\mathbf{\Lambda}_w^{2^{i+1}}$ and originate from IM α, and $\boldsymbol{S}_\beta$ be the set of CMs used by the last $K-i-1$ types of lightpaths that occupy $\mathbf{\Lambda}_w^{2^{i+1}}$ and go ahead to OM β. Note that a lightpath with granularity larger than 2^i cannot share the same CM with another one that also uses the bSlots in $\mathbf{\Lambda}_w^{2^{i+1}}$, if they share IM α or OM β. It follows that

$$\left|\boldsymbol{S}_\alpha\right| = n - p_\alpha - 1,$$
$$\left|\boldsymbol{S}_\beta\right| = n - p_\beta - 1,$$

and thus

$$\left|\boldsymbol{S}_\alpha \cup \boldsymbol{S}_\beta\right| \leq \left|\boldsymbol{S}_\alpha\right| + \left|\boldsymbol{S}_\beta\right| = 2n - 2 - \left(p_\alpha + p_\beta\right), \tag{13}$$

where the inequality holds with equality when $\boldsymbol{S}_\alpha \cap \boldsymbol{S}_\beta = \emptyset$.

$p_\alpha + p_\beta$ in (13) can be determined as follows. Any two 2^j-g lightpaths carried by p_α inputs of IM α or p_β outputs of OM β via $\mathbf{\Lambda}_w^{2^{i+1}}$ can share the same CM, as long as they do not use the same bSlot. This implies that the number of bSlots used by 2^j-g lightpaths should be larger than or equal to $2^j\left|\boldsymbol{B}_j\right|$. Also, the total number of bSlots in $\mathbf{\Lambda}_w^{2^{i+1}}$ on p_α inputs of IM α and p_β outputs of OM β is $2^{i+1}\left(p_\alpha + p_\beta\right)$. This implies

$$\sum_{j=0}^{i}\left(2^j\left|\boldsymbol{B}_j\right|\right) \leq 2^{i+1}\left(p_\alpha + p_\beta\right).$$

It follows that

$$\begin{aligned}
p_\alpha + p_\beta &\geq \left\lceil \frac{\sum_{j=0}^{i}\left(2^j\left|\boldsymbol{B}_j\right|\right)}{2^{i+1}} \right\rceil \\
&\geq \left\lceil \frac{2^i\left|\widehat{\boldsymbol{B}}_i\right| - \sum_{j=0}^{i-1}\left(2^j\left|\widehat{\boldsymbol{B}}_j\right|\right)}{2^{i+1}} \right\rceil \\
&\geq \left\lceil \frac{2^i\left|\widehat{\boldsymbol{B}}_i\right| - \sum_{j=0}^{i-1}\left(2^j\left|\boldsymbol{M}_j\right|\right)}{2^{i+1}} \right\rceil.
\end{aligned} \tag{14}$$

where we use lemma 4 for the second inequality, and (12) for the third inequality. It follows from (13) through (14) that the set of CMs that are not available for request R satisfies

$$\begin{aligned}
&\left|\widehat{\boldsymbol{B}}_i \cup \boldsymbol{S}_\alpha \cup \boldsymbol{S}_\beta\right| \\
\leq &\left|\widehat{\boldsymbol{B}}_i\right| + \left|\boldsymbol{S}_\alpha \cup \boldsymbol{S}_\beta\right| \\
\leq &\left|\widehat{\boldsymbol{B}}_i\right| + 2n - 2 - \left(p_\alpha + p_\beta\right) \\
\leq &\left|\widehat{\boldsymbol{B}}_i\right| + 2n - 2 - \left\lceil \frac{2^i\left|\widehat{\boldsymbol{B}}_i\right| - \sum_{j=0}^{i-1}\left(2^j\left|\boldsymbol{M}_j\right|\right)}{2^{i+1}} \right\rceil \\
\leq &\left|\boldsymbol{M}_i\right| + 2n - 2 - \left\lceil \frac{2^i\left|\boldsymbol{M}_i\right| - \sum_{j=0}^{i-1}\left(2^j\left|\boldsymbol{M}_j\right|\right)}{2^{i+1}} \right\rceil \\
= &2n - 1 + i(n-1) + 2n - 2 - n \\
= &2n - 2 + (i+1)(n-1).
\end{aligned} \tag{15}$$

where the fourth inequality follows from lemma 2 and (12). For arbitrary $n \geq 2$ and i, the inequality of (15) is satisfied with equality, if request R sees the following situation when it arrives:

(i) In IM α, each of $n-1$ inputs carries 2 2^0-g lightpaths and i lightpaths, the granularities of which are $2^1, 2^2, \cdots, 2^i$. All the lightpaths are allocated in $\mathbf{\Lambda}_w^{2^{i+1}}$ side-by-side in decreasing order of the granularity, so that each of them is carried by a frequency slot defined by assumption A3. IM α carries $2(n-1)$ 2^0-g lightpaths and $n-1$ 2^j-g lightpaths in total, where $j=1,2,\cdots,i$.

(ii) In OM β, one output carries a 2^0-g lightpath via $\mathbf{\Lambda}_w^{2^{i+1}}$, and each of other $n-2$ outputs carries a 2^{i+1}-g lightpath via $\mathbf{\Lambda}_w^{2^{i+1}}$. OM β carries 1 2^0-g lightpath and $n-2$ 2^{i+1}-g lightpaths in total.

All the lightpaths in (i) and (ii) are different and use different CMs, and $2n-1$ 2^0-g lightpaths are routed via the CMs in

$$\boldsymbol{B}_0 = \left\{1, 2, \cdots, 2n-1\right\},$$

and $n-1$ 2^j-g lightpaths are routed via the CMs in

$$\boldsymbol{B}_j = \left\{2n + (j-1)(n-1), \cdots, 2n - 1 + j(n-1)\right\},$$

where $j = 1, \cdots, i$, and $n-2$ 2^{i+1}-g lightpaths are routed via the CMs in

$$\boldsymbol{B}_{i+1} = \left\{2n + i(n-1), \cdots, 2n - 2 + (i+1)(n-1)\right\},$$

which is consistent with the GDR strategy.

It thus needs $m \geq 2n - 1 + (i+1)(n-1)$ CMs to satisfy R. $\square$

Fig. 7 plots a $\mathcal{C}(3,3,9)$ where 3 types of lightpaths coexist. According to the GDR strategy, $\left|\boldsymbol{M}_0\right| = 5$ and $\left|\boldsymbol{M}_1\right| = 7$. A request $R\left(2, 3, 3, 3, \mathbf{\Lambda}_1^4\right)$ sees the following situation:

(i) In IM 2, inputs 1 and 2 each carry a 2-g lightpath via $\mathbf{\Lambda}_1^2$ including bSlots 1 and 2, and 2 1-g lightpaths via $\mathbf{\Lambda}_3^2$ including bSlots 3 and 4. IM 2 carries 2 2-g lightpaths and 4 1-g lightpahts in total.

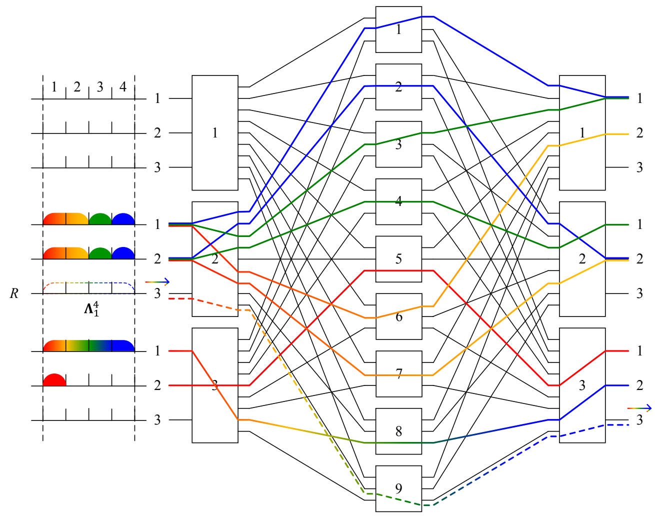

Fig. 7. $R\left(2,3,3,3,\mathbf{\Lambda}_1^4\right)$ sees the worst case in $\mathcal{C}\left(3,3,9\right)$ under the GPuB model.

(ii) In OM 3, output 1 carries a 1-g lightpath using $\mathbf{\Lambda}_1^1$, and output 2 carries a 4-g lightpath using $\mathbf{\Lambda}_1^4$.

Specifically, 5 1-g lightpaths are routed via CMs 1 through 5, 2 2-g lightpaths via CMs 6 and 7, and 1 4-g lightpath via CM 8, which is consistent with the GDR strategy. We thus need one more CM (i.e., 9 CMs in total) to satisfy R.

From the comparison of (6), (11) and (1), it is easy to show that a WSNB OXC-Clos network needs much fewer CMs than an SNB OXC-Clos network in the flex-grid scenario. Also, the GDR strategy does not introduce any routing complexity. Similar to the SNB network, the WSNB network can establish a new lightpath without reconfiguration. Therefore, the WSNB OXC-Clos network can remarkably reduce the hardware cost with little cost of operation complexity.

However, the situation is different in the fixed-grid network, where $K=1$. In this case, (6) and (11) change to $m \geq 2n-1$, indicating that the SNB and WSNB OXC-Clos networks have the same cost, which is consistent with the conclusion of [11]. This also verifies the correctness of our results.

V. Related Works

Several designs [12]–[21] have been proposed to improve the scalability of OXCs at the expense of nonblocking property. A multi-stage heterogeneous OXC was devised in [12], the idea of which is to decompose each $1\times N$ WSS in the standard OXC to a two-stage WSS structure and replace each WSS in the second stage by a wavelength-insensitive optical space switch. Another type of large-scale OXC is the interconnection of several OXCs in a ring topology [18]–[21]. Both of them are internally blocking. Applying such an internally-blocking OXC to the optical network will remarkably complicate the process of routing and spectrum allocation [8], [22].

The earliest Clos network that is able to support connections of different granularities at the same time is the multi-rate Clos network [7], [23]–[26] studied in the 1990s. The multi-rate Clos network is quite different from the flex-grid switching network. First, the former is an electrical switching network. Second, a nonblocking multi-rate Clos network can establish a connection from an input to an output, if the idle spectrum at the input and the output can provide enough bandwidth for the request. The idle spectra at the input and the output may be different and could be discontinuous. As a comparison, a nonblocking flex-grid switching network can set up a lightpath, only when the input and the output have a common continuous optical spectrum that can accommodate the request. Hence, the nonblocking conditions for multi-rate Clos networks [7], [23]–[26] cannot be applied to flex-grid Clos networks.

Refs. [27]–[29] proposed two Clos-like flex-grid switches, called space-wavelength-space (SWS) switch [27] and wavelength-space-wavelength (WSW) switch [28]. In the SWS switch, each IM and each OM are OXCs, and each CM is a bandwidth-variable wavelength-converting switch (BV-WS), which is an OXC embedded with bandwidth-variable tunable waveband converters (BV-TWBCs). In the WSW switch, each IM and each OM are BV-WSs, while each CM is an OXC. Refs. [27] and [29] derived the SNB conditions, which specify the number of CMs needed for the SWS switch and the WSW switch to achieve SNB property. However, it is known that all-optical tunable WC and BV-TWBC are not commercially available. At the current stage, it would be valuable to investigate the flex-grid Clos network without WCs.

Refs. [30]–[32] constructed large-scale OXCs based on Clos network. Specially, Ref. [30] studied the ability of OXC-Clos network with and without WCs to establish a lightpath from an input to an output if there are idle transmitters at the input and idle receivers at the output, and obtained the SNB conditions, which specify the numbers of CMs and wavelengths needed to fulfill SNB switching function. These three papers did not explore the nonblocking conditions in the flex-grid scenario.

VI. Conclusion

This paper studies the nonblocking conditions for the flex-grid OXC-Clos network without wavelength converters. The main contribution is to propose the GDR strategy and seek the WSNB conditions. The idea of the GDR is to restrict the usage of CMs by the lightpaths of different granularities, thereby leaving more CMs for large-granularity requests. Under this strategy, we prove the WSNB conditions and show that the cost of WSNB OXC-Clos network is remarkably smaller than that of SNB OXC-Clos network. We study two system models, the GPB model and GPuB model. We demonstrate that, compared to the GPB model, the GPuB model leads to more flexibility in network-bandwidth utilization only at a small cost of switching fabrics. In the future, we will explore the WSNB conditions under other granularity patterns.

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